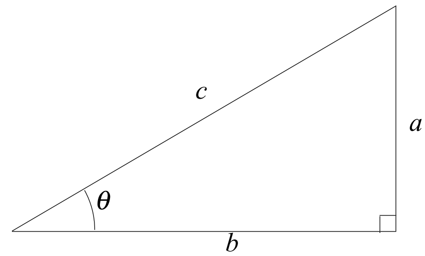
AP Physics C
Calculus

Name________________________________

Trigonometric Functions

1. Consider the right triangle to the right. In terms of *a*, *b*, and *c*, write the expressions for the following:

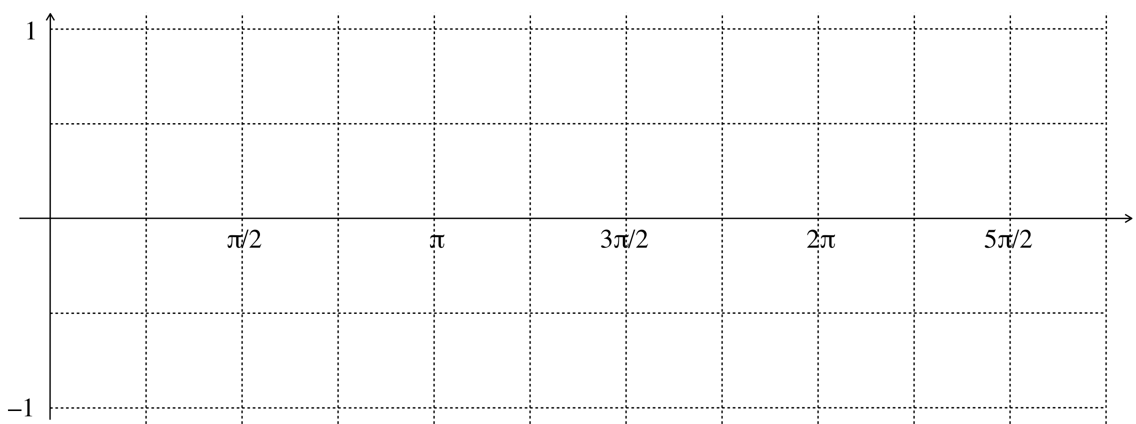


$\sin\ \theta =$ $\quad$ $\cos\ \theta =$ $\quad$ $\tan\ \theta =$

2. Using the expressions above, and the Pythagorean Theorem, verify that the following identities are true:

$\sin(90° - \theta) = \cos\theta$ $\quad$ $\cos(90° - \theta) = \sin\theta$ $\quad$ $\sin\theta / \cos\theta = \tan\theta$ $\quad$ $\sin^2\theta + \cos^2\theta = 1$

3. On the grid below, sketch and label graphs of the following functions: $y = \sin x$, $y = \cos x$, and $y = \sin(x - \pi/2)$.



Recall that the expressions for the sine and cosine of the sum of two angles are:

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi \qquad \cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$

4. Use the above expressions to derive the double angle formulas:

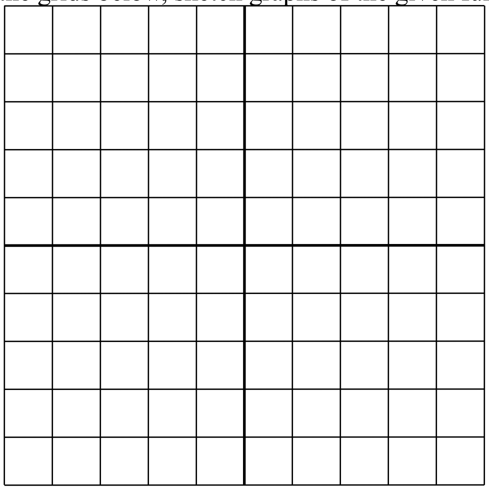
$\sin 2\theta =$ $\quad$ $\cos 2\theta =$

Exponential and Logarithmic Functions

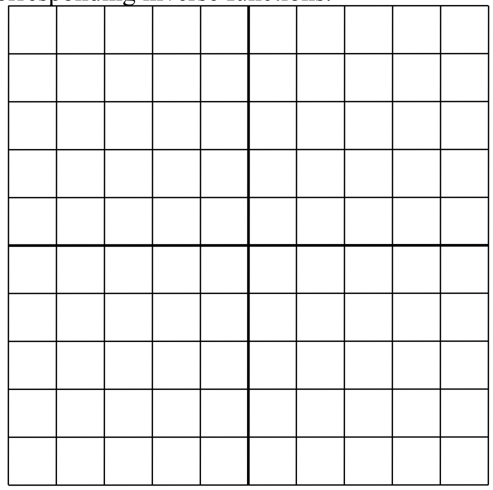
1. Use the rules for logarithms to express the following in terms of log *a* and log *b*, where *a* and *b* are positive numbers:

$\log ab =$ $\quad$ $\log (a/b) =$ $\quad$ $\log a^n =$

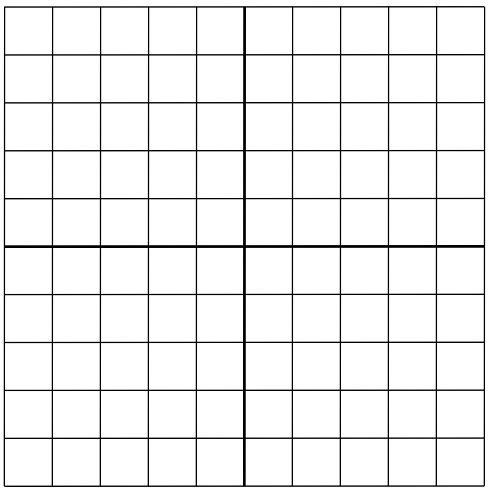
2. On the grids below, sketch graphs of the given functions and the corresponding inverse functions.



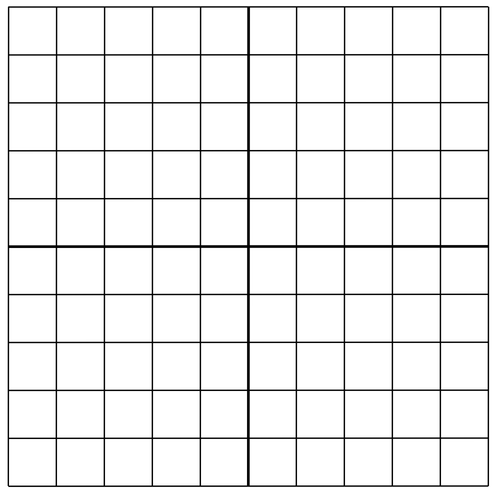
$y = 2x - 4$



inverse:

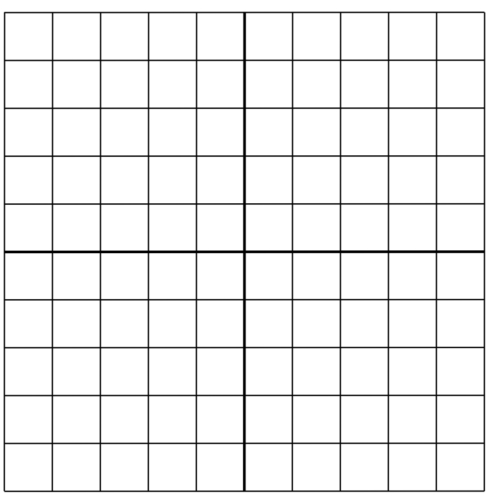


$y = x^2 - 4$

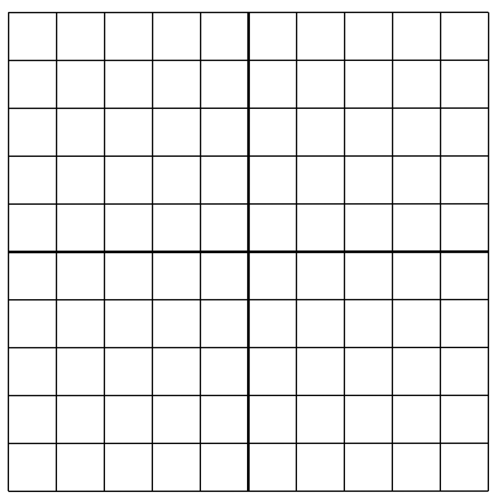


inverse:

(technically not a function)



$y = 2^x$



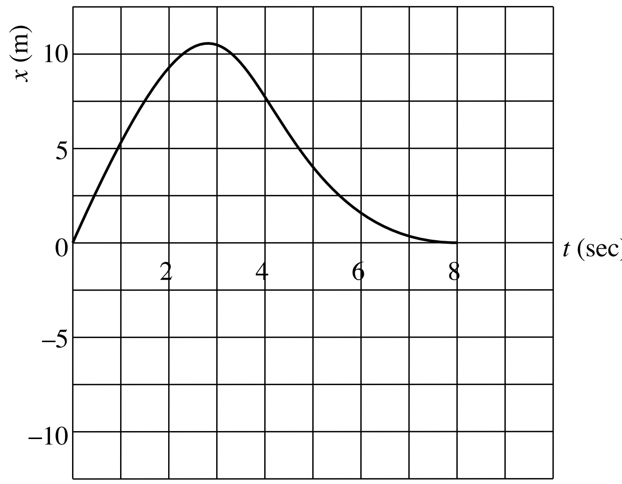
inverse:

Limits

1. Use your calculator to find approximate values of the following limits:

$$\lim_{x\to 0} \frac{\sin x}{x} = \qquad \lim_{x\to 0}(1+x)^{1/x} =$$

2. The graph to the right represents the displacement x in meters vs. time in seconds for an object moving along a line. Find the average velocity for the object for the following time intervals:



0 to 8 sec:

1 to 2 sec:

2 to 6 sec:

Describe times or time intervals for which the instantaneous velocity is:

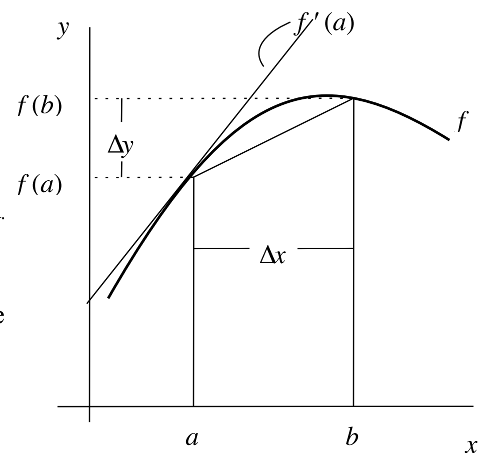
positive:

negative:

zero:

Estimate as best you can the magnitude of the greatest positive velocity and the greatest negative velocity. Explain.

Derivatives



Consider a function $y = f(x)$. The *derivative* of *f* is another function of *x*; its value at a given point is the slope of the original function at that point. This will be made much more precise in your Calculus class, but we'll attempt a slightly more precise definition here.

Consider two points on the *x* axis: *a* and *b*. The function *f* has values $f(a)$ and $f(b)$ at these points. The difference between these values is written as Δy, just as the difference between the *x* values *a* and *b* is written as Δx. The slope of the line connecting $(a, f(a))$ with $(b, f(b))$ can then be written different ways. Write the slope in terms of:

1. Δy and Δx:

2. *a, b*, and *f*:

3. *a*, Δx and *f*:

Imagine that we leave point *a* where it is and move point *b* closer to *a*. We can write this as " $b \to a$ ".

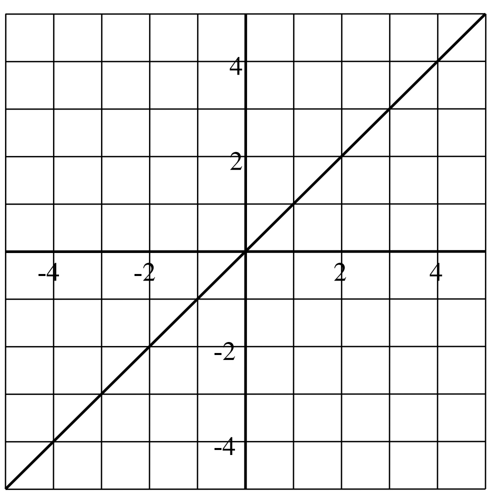
4. How would this be written in terms of Δx?

As point *b* gets closer to *a*, the slope gets closer to the slope of the line tangent to *f* at *a* (the derivative), which we write as $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ or $\frac{dy}{dx}$ for short. $\frac{dy}{dx}$ is read "*dy* by *dx*" or "the derivative of *y* with respect to *x*." *dy* and *dx* can be thought of as infinitely small versions of Δy and Δx; they are called *differentials*. The derivative of the function *f* is also written as f'.

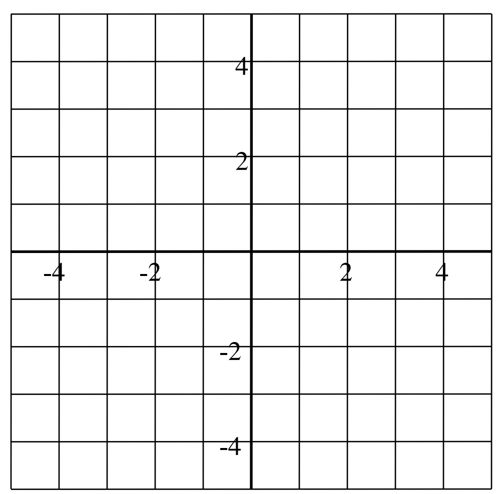
Examples

For each of the following graphs, sketch a graph of the derivative (ie., the graph of the functions whose values are the slope of the original function at each point).

1.

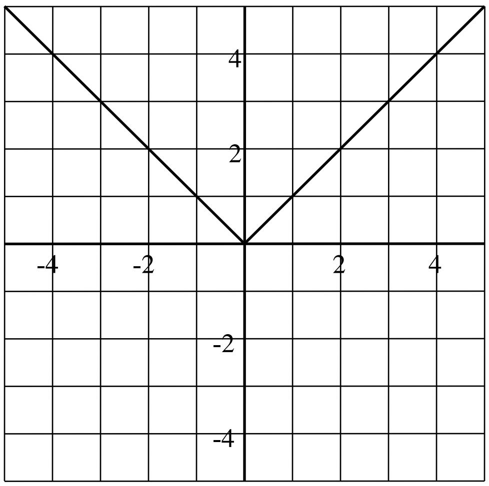


function

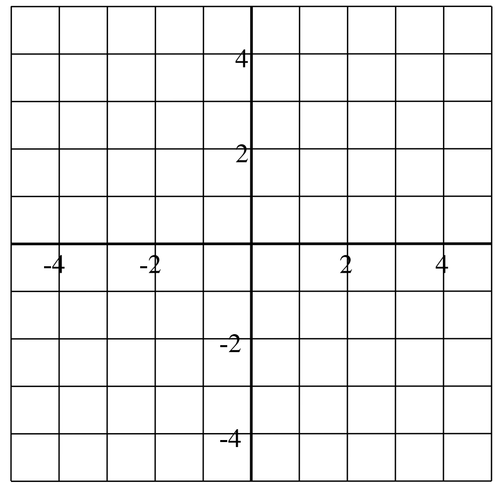


derivative

2.

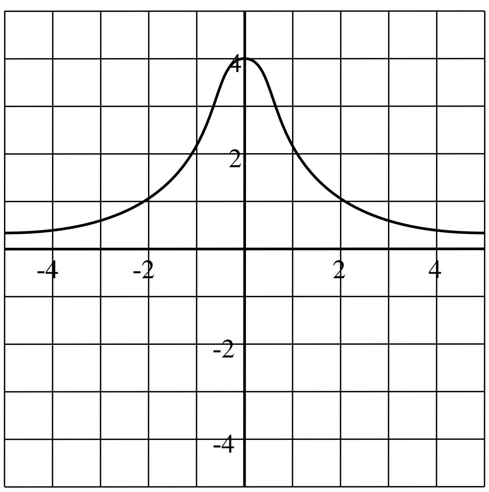


function

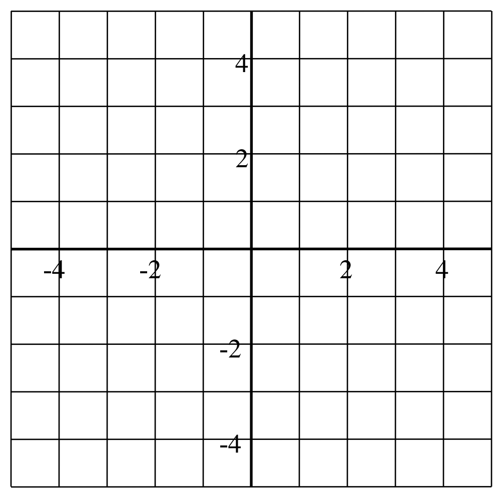


derivative

3.

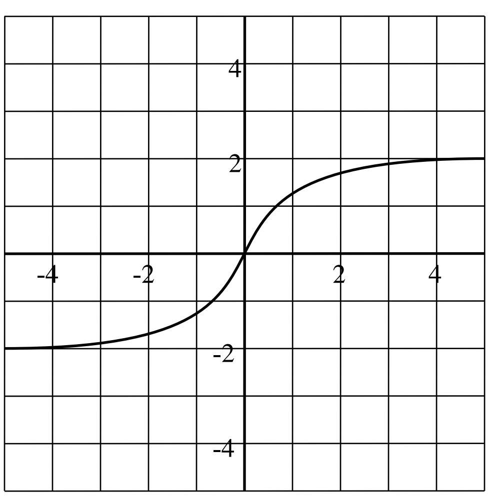


function

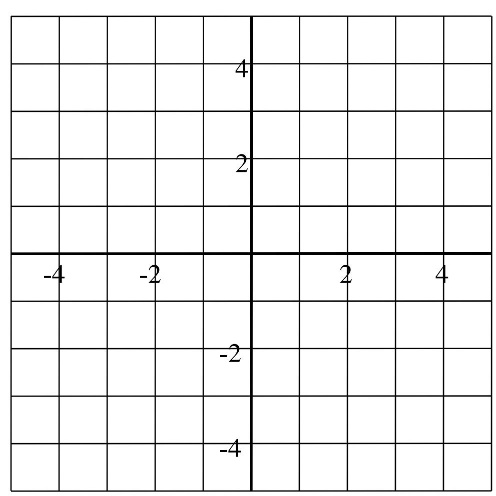


derivative

4.

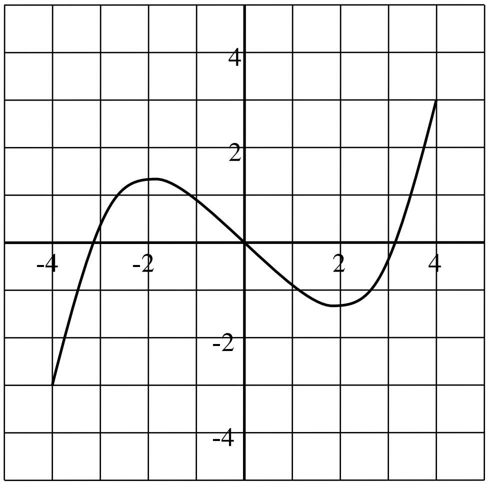


function

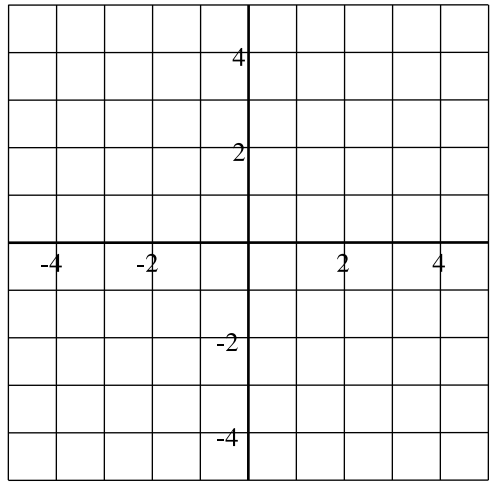


derivative

5.



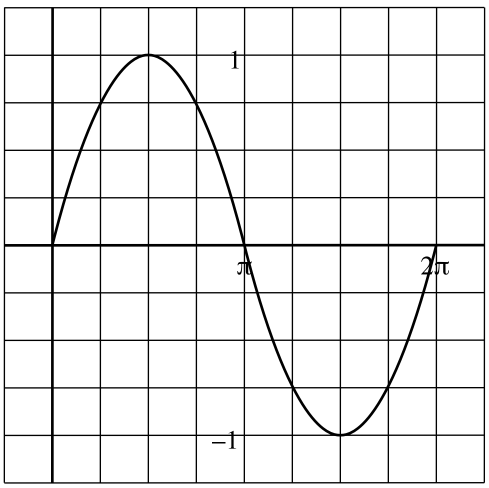
function



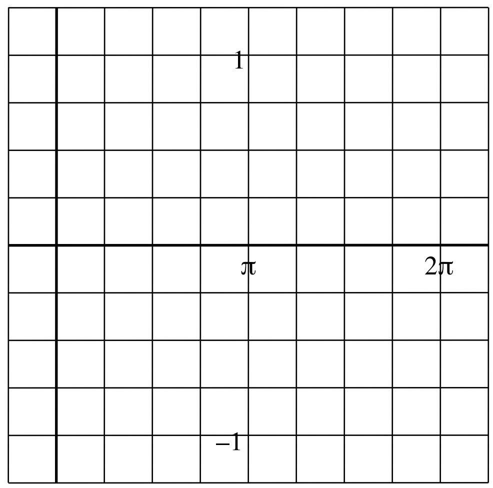
derivative

For the following, guess the derivative function after sketching its graph. Note that for the next two, the vertical and horizontal axes are not scaled the same.

6.

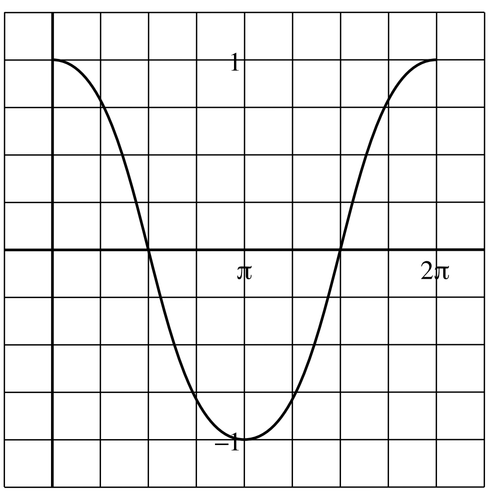


function: $y = \sin x$

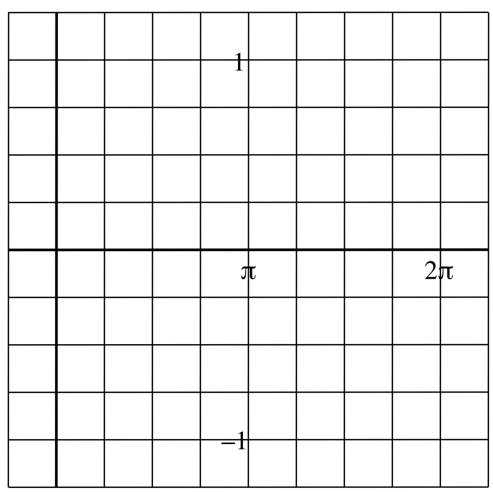


derivative:

7.

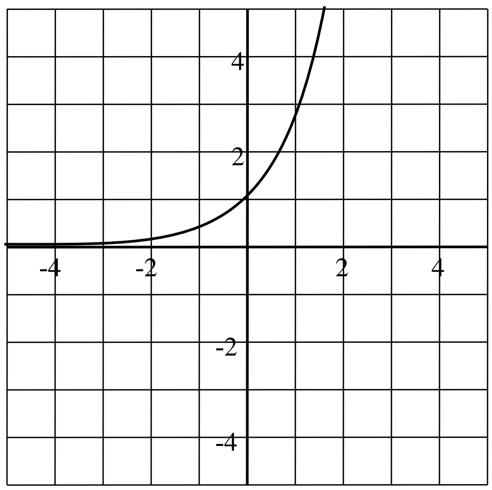


function: $y = \cos x$

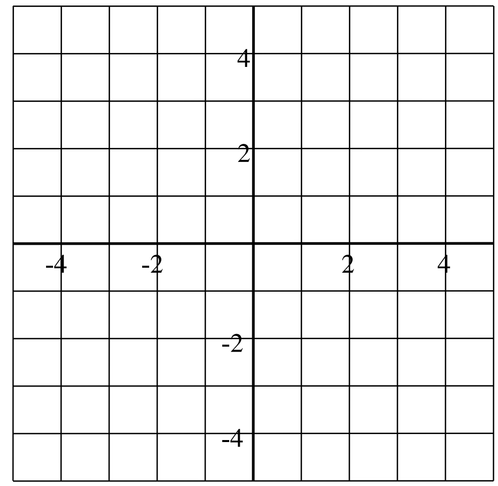


derivative:

8.

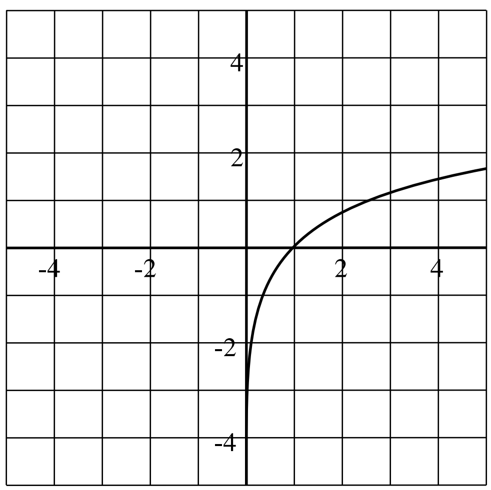


function: $y = e^x$

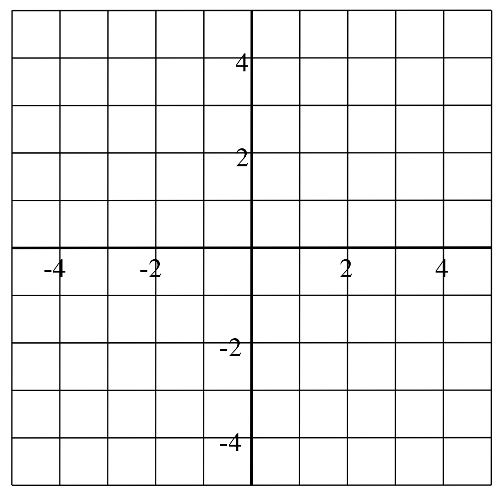


derivative:

9.



function: $y = \ln x$



derivative:

Differentiation

The process of finding the derivative of a function is called *differentiating* the function (not "deriving" the function). The functions you need to be able to differentiate are the power functions (polynomials), trigonometric functions (sin, cos and tan), and e^x and ln *x*, and functions built from these. In he rules for differentiation to the right, *u* and *v* stand for functions of *x*, and *a* and *m* are constants.

Although it's not in the list to the right, we might consider rule zero:

0. $\frac{da}{dx} = 0$

That is, the derivative of a constant function is zero. Explain why this is so:

1. $\frac{dx}{dx} = 1$
2. $\frac{d}{dx}(au) = a\frac{du}{dx}$
3. $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$
4. $\frac{d}{dx}x^m = mx^{m-1}$
5. $\frac{d}{dx}\ln x = \frac{1}{x}$
6. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
7. $\frac{d}{dx}e^x = e^x$
8. $\frac{d}{dx}\sin x = \cos x$
9. $\frac{d}{dx}\cos x = -\sin x$
10. $\frac{d}{dx}\tan x = \sec^2 x$

In addition to the list on the previous page, you need to know the following rules for breaking down complex functions:

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$ (called the *quotient rule*: "Lo d-hi minus hi d-lo over lo-lo")

$$\frac{d}{dx}\big(u(v(x))\big)=\frac{du}{dv}\frac{dv}{dx}$$ (called the *chain rule*)

The chain rule will be used very frequently to differentiate functions like $y=\sqrt{1+x^2}$, where $v(x)=1+x^2$ and $u(v)=\sqrt{v}$.

Examples

Refer to the table of derivatives differentiate the following functions:

1. $y=x^4+8x^3 \quad y'=\frac{dy}{dx}=$

2. $y=x^2x^3 \quad y'=$ A) B)

(Do this two ways: A) multiply out and B) use the product rule)

3. $y=(x^5+7)(3x^2+1) \quad y'=$ A) B)

(Do this two ways: A) multiply out and B) use the product rule)

4. $y=\frac{1+x}{x^2} \quad y'=$ A) B)

(Do this two ways: A) divide out and B) use the quotient rule)

5. $y=\sqrt{1+x^2} \quad y'=$

(Use the chain rule)

6. $y=\frac{\sqrt{x}-1}{\sqrt{x}+1} \quad y'=$

7. $y=\cos(2x) \quad y'=$

8. What angle does the function above make with the *x*-axis?

9. $y = \sin(3x) + \cos(2x^2)$ $y' =$

10. $y = \sin^2(3x - 2)$ $y' =$

For the following, find the higher order derivatives:

11. $y = 2x^3$ $y'' = \dfrac{d^2y}{dx^2} =$

12. $y = x + \dfrac{1}{x}$ $y'' =$

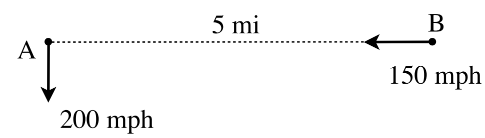
13. $y = x^4$ $\dfrac{d^4y}{dx^4} =$

14. $y = x^2 e^x$ $y'' =$

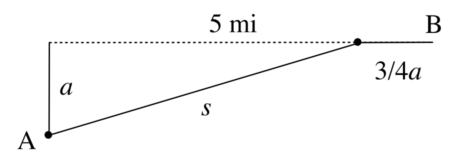
Minima and Maxima

Often a function reaches a local maximum (like the function *f* on page C.3 just before *b*) or a local minimum. The slope of the function (its derivative) would be zero there. We use this to solve problems like the following:

At a certain instant, two airplanes A and B are 5 mi apart. A is due west of B going at 200 mph due south, and B is going 150 mph due west. How close will these planes get to each other?



To solve the problem, express the distance between the planes as a function of some other variable, differentiate the function, and find when the derivative is zero. For example, let plane A cover a little distance *a*. In that time, B will cover 3/4 that distance because it's going 3/4 as fast.



First, use the Pythagorean Theorem to express the distance *s* between the planes as a function of *a*.

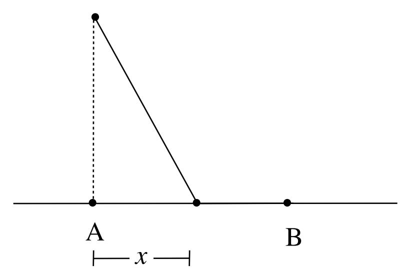
Now find the derivative of *s* with respect to *a* $\left(\dfrac{ds}{da}\right)$.

Set the derivative equal to zero and solve for *a*; this will tell you how far plane A has gone by the time *s* is minimized.

Then use this value to find how close they are.

Examples

1. A kayaker finds herself one mile out from a long, straight beach. She wishes to get to a point B on the beach that's one mile along the beach from the point closest to her (point A) as fast as possible. She can kayak at 4 mph, and jog along the beach at 8 mph. Her plan is to paddle to a point a certain distance x from A, leave the kayak and jog the rest of the way.

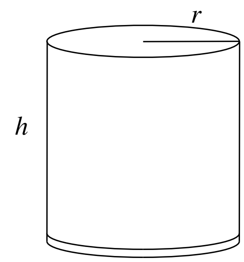


A) What value of x will minimize the time it takes?
(You must express the time t in terms of x, take the derivative of t with respect to x (dt/dx), set it equal to zero and solve for x.)

B) How long will it take her?

2. A glassblower wishes to blow a thin, cylindrical tumbler. The bottom of the tumbler is to be three times as thick as the sides. If the tumbler is to hold a volume *V*, what are the dimensions (radius and height) that will minimize the amount of glass it takes?

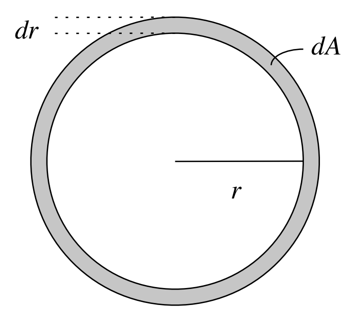
(Think of the sides of the tumbler as a rectangle, and the bottom as three circles. Express the "area" *A* of glass as a function of *r*, and find the value of *r* that minimizes *A*. Then find the corresponding value of *h*. Both *r* and *h* should be expressed in terms of *V*).



Differentials

dx is a *differential*, or an infinitesimal (very small) change in *x*. For example, if $y = 3x^4$, then $\frac{dy}{dx} = 12x^3$. We can treat the *dy* and the *dx* as separate things, and solve for *dy*: $dy = 12x^3 dx$. This is called a *differential equation*; that is, an equation involving differentials. Another way to say this is that if $y = 3x^4$, then when *x* changes by a differentially small amount (*dx*) then *y* changes by $12x^3$ times as much.

The advantage of differentials is that when things get very small they get simple. For example, consider a circle of radius *r* as shown to the right. Imagine increasing its radius by a differentially small amount *dr*. By how much has the area *A* of the circle changed?



Since *dr* is so small, this change in area *dA* can be thought of as a rectangle.

What are the dimensions of this "rectangle"? Length: Width:

Write the differential equation for the area *dA*:

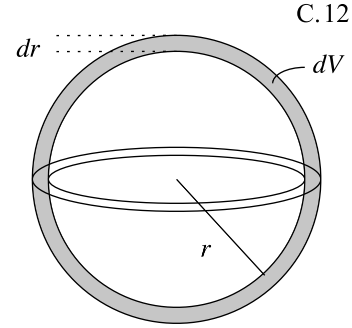
The "differential equation" description of this increase in area is equivalent to another "derivative" version.

What is the area *A* of the circle? $A =$

What is the derivative of *A* with respect to *r*? $\frac{dA}{dr} =$

Solve this derivative equation for *dA* to get a differential equation:

Now fill in the blanks to make the analogous statements for a sphere of radius r, considering a differential increase in the volume:



Imagine increasing its radius by a small amount dr. By how much has the volume V of the sphere changed?

Since dr is so small, this change in volume dV can be thought of as a solid shape.

What are the dimensions of this "solid shape"? Area: [ ] Height: [ ]

Write the differential equation for the volume dV: [ ]

The "differential equation" description of this increase in volume is equivalent to another "derivative" version.

What is the volume V of the sphere? $V =$

What is the derivative of V with respect to r? $\frac{dV}{dr} =$

Solve this derivative equation for dV to get a differential equation: [ ]

Integrals

Consider a function *f(x)*. We define the *antiderivative* of f as another function F whose derivative is f; ie., $\frac{dF}{dx} = f$. For example, if $f(x) = x^n$, then except when $n = -1$, $F(x) = \frac{x^{n+1}}{n+1}$. Explicitly show why this follows from rule #4 on page C.7:

[ ]

Since the derivative of a constant is zero, specifying f doesn't uniquely determine F, because you can freely add constants to F and its derivative is still f. We use the term *integral* instead of antiderivative, and use the following notation:

$$\int f(x)\,dx = F(x) + c$$

to mean "the integral of f of x with respect to x is F of x." c is an arbitrary constant called the *constant of integration*. Often the constant is understood, not explicitly stated. For example, the expression:

$$\int (\ln x)\,dx = x\ln x - x + c$$

means: The function you take the derivative of to get the function "$\ln x$" is the function "$x\ln x - x$". Use the rules for differentiation to show this:

[ ]

Refer to the list of integrals to the right. *You should know these integrals by heart. Take the time now to study this list of integrals!*

1. $\int dx = x$
2. $\int au\, dx = a\int u\, dx$
3. $\int (u+v)\, dx = \int u\, dx + \int v\, dx$
4. $\int x^m\, dx = \dfrac{x^{m+1}}{m+1}\ (m \neq -1)$
5. $\int \dfrac{dx}{x} = \int \dfrac{1}{x}\, dx = \ln|x|$
6. $\int e^x\, dx = e^x$
7. $\int \sin(x)\, dx = -\cos(x)$
8. $\int \cos(x)\, dx = \sin(x)$
9. $\int e^{-ax}\, dx = -\dfrac{1}{a}e^{-ax}$

Note that when you take the derivative of the function on the right, you should get the function that appears between the integral sign and the differential.

For #1, taking the derivative of x with respect to x gives 1, and there is an understood 1 on the left, so that it could read $\int (1)\,dx = x$.

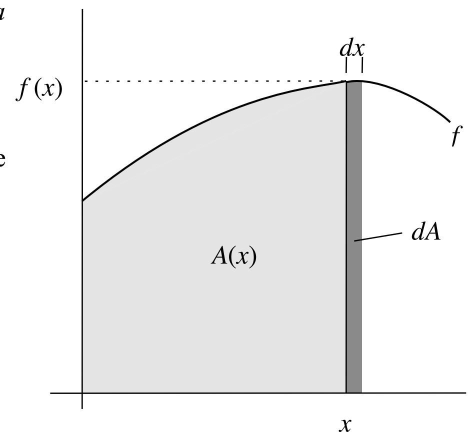
#3 means that polynomials can be integrated term by term.

#5 takes care of the exception to #4.

#6 is a special case of #9 #9 will be used *very* frequently in this course.

Integrals and Area

Consider an arbitrary function $f(x)$. Define a new function of x, we'll call the *area* function A, as follows: For a given x, find the area under the $f(x)$ curve from the y-axis to a vertical line at x. Area below the x-axis is considered negative. We call this area function $A(x)$.



Now consider increasing x by dx, as we did on page C.11. By how much does the area function increase (dA)?

$dA =$

So then, using integral notation, write an expression for the function A:

$A =$

So just as the derivative of f is the slope of its graph, the integral of f is the area under its graph.

Integration by Change of Variable

Note that although the integral of a sum is the sum of the integrals (#3 in the list above), *there is no product rule, quotient rule or chain rule for integration*. There are various techniques for integrating products, quotients and compositions of functions, one of which is called *change of variable*. The steps are these:

1. Pick a new variable (such as u) to represent a portion of the integrand.
2. Find the differential of u (du).
3. Re-express the integrand in terms of u and du.

If this process results in an integral you can evaluate, then do so, otherwise you may have to try again from step 1, letting u represent some other portion of the integrand.

For example, consider finding $\int \dfrac{x}{x^2+4}dx$: Let $u = x^2 + 4$. Then what's du? (Hint: find $\dfrac{du}{dx}$ and solve for du.)

Now rewrite the original integral in terms of u and du, with no x's or dx's. Use this to find the integral in terms of u, then substitute back again.

Integration by Separation of Variables

Recall that a *differential equation* is simply an equation involving differentials, such as $dx = x^2 dt$. The solution of a differential equation is a function; in this case $x(t)$. To solve such an equation (find the function *x*):

A) separate the variables such that all *x*'s occur on the same side as *dx*, and all *t*'s (if there were any) are on the same side as *dt*.

B) Take the integral of both sides.

C) Use the table of integrals to simplify both sides (ignore the integration constants).

D) What is the function $x(t)$?

E) Show that this function satisfies the differential equation $dx = x^2 dt$.

Examples

Find the following integrals by change of variable. No need to specify an integration constant (as we will see, we won't need integration constants in physics). Every solution should start with "let *u* = ..., then *du* = ..." and go from there.

1. $\int \frac{dx}{1+x}$

2. $\int \sin^2\theta \cos\theta \, d\theta$

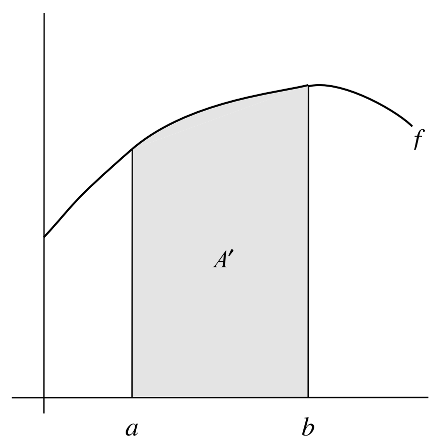
3. $\int x\sqrt{1+x^2}\, dx$

4. $\int \sin 3x \, dx =$

5. $\int \frac{8x^2 \, dx}{(x^3+2)^3} =$

6. $\int xe^{-x^2} \, dx =$

The Definite Integral



Consider the area function *A* defined above. Now we define a new area function *A′*, as the area under *f*(*x*) between two particular values of *x*; *a* and *b*. Clearly this new area function is the original area function at *b* minus the original area function at *a*: *A′* = *A(b)* – *A(a)*. This new function is called the *definite integral of f between a and b*, and *a* and *b* are called the *limits of integration*. If *F* is the antiderivative of *f*, then the definite integral is written:

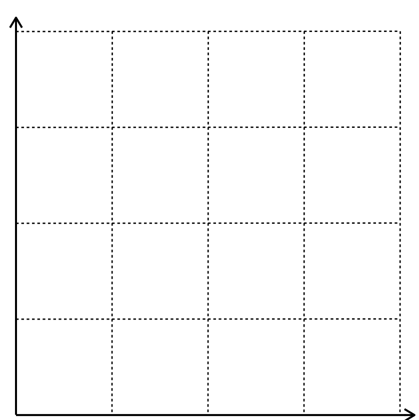
$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $F(x)\Big|_a^b$ is just shorthand for evaluating *F* at the limits of integration, and taking the difference.

Here's an example: Find the area under the curve $y = x^2$ between $x = 1$ and $x = 2$.

The solution:

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$$

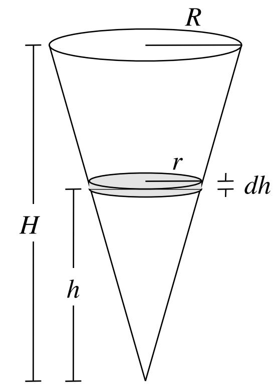


Sketch a graph of the function to the right and see whether this result is in the ballpark.

In physics we will always use the definite integral. Explain why we never need to specify the integration constant if we use the definite integral.

Applications of Integration

Integration can be used to find areas and volumes of various shapes. For example, find the volume of a cone of height H and base radius R as follows:



Consider a thin slice of the cone; let the variable h represent the distance of this slice from the vertex. Then the thickness of the slice is a small change in h: dh.

Use similar triangles to find the radius r of the slice in terms of R, H, and h:

The slice has a differentially small volume dV. The thickness of the slice is so small that the sides can be considered vertical, and the radius of the bottom is also r. What is the volume dV of the slice in terms of R, H, and h and the thickness dh?

This is a differential equation. We solve this equation for V by integrating, this time using the definite integral. To find the limits of integration, imagine what range of values h has to have for the slice to sweep over the entire volume we're looking for; these are the two limits. Write the definite integral we need to evaluate:

Evaluate this definite integral to find V.

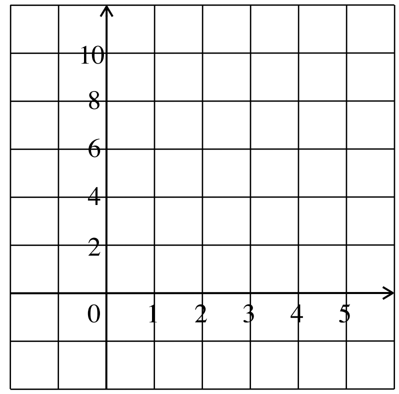
Examples

1. Find the area under the curve $y = \sin x$:
   A) from 0 to π,

   B) from 0 to $\pi/2$,

   C) from 0 to 2π.

2. Find the area bounded by the curves $y = 6x - x^2$ and $y = x^2 - 2x$.

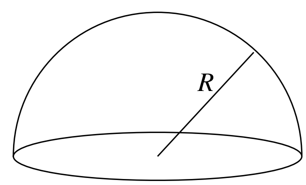


A) Make a sketch of the two functions to the right.
B) *Describe* (in words) how you can use definite integrals to find the area bounded by these two curves. Be specific about the limits of integration.

C) Translate your description into an expression involving the corresponding definite integrals.

D) Simplify this expression as much as possible *before* integrating, then perform the integration.

3. Find the volume of a hemisphere of radius R by dividing it into thin disks and integrating their volumes (similar to the cone on page C.16). *You must clearly define, and draw in the diagram, any variables you need.*



Exercises

Finding Derivatives

1. $y = 2x^{1/2} + 6x^{1/3} - 2x^{3/2}$ $\qquad \dfrac{dy}{dx} =$

2. $s = (t^2 - 3)^4$ $\qquad \dfrac{ds}{dt} =$

3. $f(t) = (2t-1)\sqrt{3-t^2}$ $\qquad \dfrac{df}{dt} =$

4. $x = y\sqrt{1-y^2}$ $\qquad \dfrac{dx}{dy} =$

5. $y = \tan x^2$ $\qquad \dfrac{dy}{dx} =$

6. $y = \tan^2 x$ $\qquad \dfrac{dy}{dx} =$

7. $f(x) = x^2 \sin x$ $\qquad \dfrac{df}{dx} =$

8. $y = \ln(3x^2 - 5)$ $\qquad \dfrac{dy}{dx} =$

9. $y = \ln(x+3)^2$ $\qquad \dfrac{dy}{dx} =$

10. $y = \ln^2(x+3)$ $\qquad \dfrac{dy}{dx} =$

11. $y = \ln(\sin 3x)$ $\qquad \dfrac{dy}{dx} =$

12. $f = \ln(\sin\theta)$ $\qquad \dfrac{df}{d\theta} =$

13. $y = x\ln(x) - x$ $\qquad \dfrac{dy}{dx} =$

14. $y = \sqrt{1+x^2}$ $\qquad \dfrac{dy}{dx} =$ $\qquad \dfrac{d^2y}{dx^2} =$

15. $f = \tan\theta$ $\qquad \dfrac{df}{d\theta} =$ $\qquad \dfrac{d^2f}{d\theta^2} =$

16. $y = xe^{x^2}$ $\qquad \dfrac{dy}{dx} =$ $\qquad \dfrac{d^2y}{dx^2} =$

17. Find the slope of the curve $x = y^2 - 4y$ at the points where it crosses the y-axis.

18. Find the equation for the line tangent to the curve $y = \dfrac{\ln x}{x}$ at $x = \sqrt{e}$.

19. A point moves along the curve $y = x^3 - 3x + 5$ so that $x = \dfrac{\sqrt{t+3}}{2}$, where t is time. At what rate is y changing when $t = 6$ (in arbitrary units)?

20. Apple Computer Inc. is coming out with a new iPod model, and they wish to maximize the gross income from its sales. If N iPods are sold at a price p, the gross income is Np. However, as the price is raised, the number of buyers decreases. It is estimated that the number of buyers at price p can be described by the expression to the right, where N_o and p_o are constants. Note that as the price approaches p_o the sales approach zero; the expression is meaningless for $p > p_o$. What should be the price for the maximum income, and what is the maximum income?

$$N(p) = N_o\left[1 - \left(\frac{p}{p_o}\right)^2\right]$$

Indefinite Integrals

21. $\int \frac{dx}{x^{2/3}} =$

22. $\int (2x^2 - 5x + 3)\, dx =$

23. $\int (1 - x)\sqrt{x}\, dx =$

24. $\int \frac{x^2}{\sqrt[4]{x^3 + 2}}\, dx =$

25. $\int \frac{\ln x}{x}\, dx =$

26. $\int \sin x\, e^{\cos x}\, dx =$

Definite Integrals

27. $\int_0^3 x^2\, dx =$

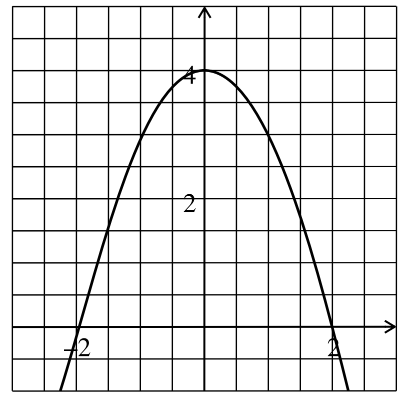
28. $\int_{-1}^1 \left(2x^2 - x^3\right) dx =$

29. $\int_1^2 (2 - 3x)^3\, dx =$

30. $\int_0^4 \frac{1}{x+5}\,dx =$

31. $\int_{-1}^{1} \frac{e^x}{1+e^x}\,dx =$

32. Find the area bounded by the curve $y = -x^2 + 4$ and the x-axis.



33. Find the volume of a pyramid with a square base of side L and a perpendicular height of H.